## Scaling Properties, Fractals, and the Renormalisation Group Approach to Percolation

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## **Article Outline**

#### Glossary

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## Glossary

#### Cluster

Clusters are sets of occupied neighbouring sites.

#### Critical exponents

At a critical point or second-order phase transition, many quantities diverge or vanish with a power law of the distance from this critical point; the critical exponent is the exponent for this power law.

### Fractals

Fractals have a mass varying with some power of their linear dimension. The exponent of this power law is called the fractal dimension and is smaller than the dimension of the space.

#### Percolation

Each site of a large lattice is randomly occupied or empty.

### Renormalization

A cell of several sites, atoms, or spins is approximated by one single site etc. At the critical point, these supersites behave like the original sites, and the critical point thus is a fixed point of the renormalisation.

## 1 Definition and Introduction

Paul Flory, who later got the Chemistry Nobel prize, published in 1941 the first percolation theory [22], to describe the vulcanisation of rubber [1]. Others later applied and generalised it, in particular by dealing with percolation theory on lattices and by studying it with computers. Most of the theory presented here was known around 1980, though in the case of computer simulation with less accuracy than today. But on the questions of universality, of critical spanning probability and of the uniqueness of infinite clusters, the 1990's have shown some of our earlier opinions to be wrong. And even today it is questioned by some that the critical exponents of percolation theory can be applied to real polymer gelation, the application which Flory had in mind two-thirds of a century ago.

On a large lattice we assume that each site independently and randomly is occupied with probability p and empty with probability 1-p. Depending on applications, also other words can be used instead of occupied and empty, e.g. Republican and Democrat for the majority party in an electoral district of the USA. A *cluster* is now defined as set of occupied neighbouring sites. Percolation theory deals with the number and structure of these clusters, as a function of their size s, i.e. of the number s of occupied sites in the cluster. In particular it asks whether an infinite cluster spans from one side of the lattice to the opposite side. Alternatively, and more naturally if one wants to describe chemical reactions for rubber vulcanisation, this site percolation can be replaced by bond percolation, where every site is occupied but the link between neighbouring sites is either present with probability p or absent with probability 1-p, again independently and randomly for each link. A cluster is now a set of neighbouring sites connected by links, and the size s of the cluster can be counted as the number of links, or as the number of sites, in that cluster. Because of this ambiguity we discuss here mainly site percolation; bond percolation is similar in the sense that it belongs to the same universality class (same critical exponents). One may also combine both choices and study site-bond percolation where each site is randomly occupied or empty, and where each bond between neighbouring occupied sites is randomly present or absent.

Neither temperature nor quantum effects enter this standard percolation model, which is purely geometrical probability theory. However, to understand why percolation works the way it does it is helpful to understand thermal phase transitions like the vapour-liquid critical point; and for magnetic applications it is useful to know that some spins (atomic magnetic moments) have only two states, up or down, according to quantum mechanics. We will explain these physics aspects later.

For small p, most of the occupied sites are isolated s=1, coexisting with only few pairs s=2 and triplets s=3. For large p, most of the occupied sites form one "infinite" cluster spanning the lattice from left to right, with a few small isolated holes in it. Thus there exists one percolation threshold  $p_c$  such that for  $p < p_c$  we have no spanning cluster and for  $p > p_c$  we have (at least) one spanning cluster. Inspite of decades of research in this seemingly simple problem, no exact solution for  $p_c$  is proven or guessed for site percolation on the square lattice with nearest-neighbour bonds; only numerically we know it to be about 0.5927462. For site percolation on the triangular lattice or bond percolation on the square lattice,  $p_c = 1/2$  exactly. More thresholds are given in Table 1 [22]. They are valid in the limit of  $L \to \infty$  for lattices with  $L^d$  sites in d dimensions. For small L instead of a sharp transition at  $p_c$  one has a rounded changeover: with a very low probability one chain of L occupied sites at  $p = 1/L^{d-1}$  spans from left to right. In one dimension, a small chain can easily be spanned if p is close to one, but for  $L \to \infty$  the threshold approaches  $p_c = 1$  since at smaller p a hole will appear about every 1/(1-p) sites and prevent any cluster to span.

## 2 Methods

This section summarises some of the methods employed to find percolation properties, first by pencil and paper, and then with the help of computers for which Fortran programs are published e.g. in [3, 4]. More details on simulations are reviewed by Ziff in this percolation part of this encyclopedia.

#### 2.1 Mean field limit

The Bethe lattice or Cayley tree neglects all cyclic links and allows a solution with paper and pencil. We start from one central site, and let z bonds emanate from that. At the end of each bond sits a neighbour. Then from each of these neighbours again z bonds emanate, one back to the central site and z-1 to new sites further outward. They in turn lead again each to z-1 new sites, and so on. None of the newly added sites agrees with one of the already existing sites, and so we can travel along the bonds only outwards

$p_c$	site	bond
d=1 chain	1	1
honeycomb	.697043	$1 - 2\sin(\pi/18)$
square	.592746	1/2
triangular	1/2	$2\sin(\pi/18)$
diamond	.4301	.3893
SC	.311608	.248813
BCC	.245691	.180287
FCC	.199236	.120163
d = 4 hypercubic	.196885	.160131
d = 5 hypercubic	.140797	.118172
d = 6 hypercubic	.109018	.094202
d = 7 hypercubic	.088951	.078675

Table 1: Site and bond percolation thresholds for one dimension, three twodimensional, four three-dimensional and four hypercubic lattices in higher dimensions. [22, 2]

or back, but never in a circle. It is quite plausible that an infinite cluster of bond percolation is formed if each site leads to at least one more outward site along an existing bond, that means if (z-1)p > 1. This condition also holds for site percolation. Thus

$$p_c = 1/(z-1)$$
 . (1)

In this way Flory calculated the threshold and other percolation properties. Today we call this the "mean field" universality class in analogy with thermal phase transitions. The critical exponents, to be discussed below, are integers or simple fractions. To this universality class belong also the Erdös-Rényi random graphs, where we connect in an assembly of N points each pair with a low probability  $\propto 1/N$ . And the same universality class is reached if we let the dimension d of the hypercubic lattice go to infinity (or at least take it above 6). A disadvantage of the Bethe lattice is its lack of realism: If the length of the bonds is constant, then the exponential increase of the number of sites and bonds with increasing radius leads to an infinite density.

#### 2.2 Small clusters

The probability of a site to be an isolated s = 1 cluster on the square lattice is  $n_1 = p(1-p)^4$  since the site must be occupied and all its four neighbours be empty. The formula for pairs is  $n_2 = 2p^2(1-p)^6$  since the pair can be oriented horizontally or vertically, resulting in the factor 2. Similar, only more difficult, is the evaluation of  $n_s$  with a maximum s usually 10 to 20; the general formula is

$$n_s = \sum_t g_{st} p^s (1 - p)^t \tag{2}$$

where the perimeter t is the number of empty neighbours and  $g_{st}$  is the number of configurations (or lattice animals, or polyominoes) of size s and perimeter t. The King's College group in London published these results decades ago. With techniques borrowed from series expansions near thermal critical phenomena, these polynomials allow to estimate not only  $p_c$  but also many other quantities (see below) diverging or vanishing near  $p_c$ .

## 2.3 Leath cluster growth

In the cluster growth method of Leath (1976) one starts with one occupied site in the centre of the lattice. Then a cluster is grown by letting each empty neighbour of an already occupied cluster site decide once and for all, whether is is occupied or empty. One needs to keep and to update a perimeter list of empty neighbours. If that list becomes empty, the cluster growth is finished, and no boundary effects of the lattice influence this cluster. If, on the other hand, the cluster reaches the lattice boundary, one has to stop the simulation and can regard this cluster as spanning (from the centre to one of the sides). Repeating many times this growth simulation one can estimate  $p_c$  as well as the cluster numbers. More precisely, the cluster statistics obtained in this way is not  $n_s$  but  $n_s s$  since the original centre site belongs with higher probability to a larger than to a smaller cluster.

# 2.4 Hoshen-Kopelman labelling

To go regularly through a large lattice, which may even be an experimentally observed structure to be analysed by computer, one could number consecutively each seemingly new cluster, and if no clusters merge later then one has a clear classification: All sites belonging to the first cluster have label 1,

all sites of the second cluster have label 2, etc. Unfortunately, this does not work. In the later analysis it may turn out that two clusters which at first seemed separate actually merge and form one cluster:

Already in the simple structure shown on the left we have several such label conflicts. The labels to the right come from going though the lattice like a typewriter, from left to right, and after each line to the lower line. When we come to the right neighbour of the 3 we see that 3 is really part of the cluster with label 2. And at the right neighbour of 4 we see that 4 belongs to cluster 1. The stupid method is to go back and to relabel all 3 into 2, and all 4 into 1. If then we come to the site marked with x we see that the whole structure is really one single cluster, and thus all labels 2 have to be relabeled into a 1. This is inefficient for large lattices. Instead, Hoshen and Kopelman (1976) gave each site label  $m = 1, 2, 3, \ldots$  another index n(m). This label n(m) of labels equals its argument, n(m) = m, if it is still a good "root label", and it equals another number k is the cluster with initial label m later turned out to be part of an earlier cluster k. By iterating the command m = n(m) until finally the new m equals n(m) one finds this root label. For the above we make the following assignments and re-assignments to n: n(1) = 1, n(2) = 2, n(3) = 3, n(3) = 2, n(4) = 4, n(4) = 1, n(2) = 1. Clusters are now characterised by the same root label for all their labels.

An advantage if this method is that only one line of the square lattice, or one hyperplane of the d-dimensional lattice, needs to be stored at any time, besides the array n(m). And that array can also be reduced in size by regular recycling no longer used labels n, just as beer bottles can be recycled. Lattices with more than  $10^{13}$  sites were simulated, using parallel computers. However, understanding the details of the algorithms and finding errors in them can be very frustrating.

Sometimes one wants to determine the cluster numbers for numerous different p from 0 to 1. Instead of starting a new analysis for each different p one may also fill the lattice with new sites, and make the proper labelling of labels whenever a new site was added [5]. Similarly, one can determine the properties of various lattice sizes L by letting L grow one by one and relabeling the cluster after each growth step [6]. Unfortunately, these two

methods came long after most of the percolation properties were already studied quite well by standard Hoshen-Kopelman analysis.

## 2.5 Relation to Ising and Potts models

The relation between percolation and thermal physics was useful for both sides: Scaling theories for percolation could follow scaling theories for thermal physics from ten years earlier, and computer simulations for thermal physics could use the Leath and Hoshen-Kopelman algorithms of cluster analysis, leading to the Wolff and Swendsen-Wang methods, respectively, a decade later. A mathematical foundation is given by the Kasteleyn-Fortuin theorem [8] for the partition function Z of the Q-state Potts model at temperature T:

$$Z(Q) = \langle Q^N \rangle \tag{3}$$

where N is the total number  $\sum_s n_s$  of clusters for bond percolation at probability  $1 = \exp(-2J/k_BT)$ ,  $\langle \ldots \rangle$  indicates an average over the configurations at this probability,  $k_B$  is Boltzmann's constant and 2J is the energy needed to break a bond between neighbouring spins.

Q values of 3 and larger are interesting since for increasing Q a second-order phase transition with a continuous order parameter changes into a first-order phase transition with a jumping order parameter, when T increases. The special case Q=2 is the spin 1/2 Ising model (the model is pronounced EEsing, not EYEsing since Ernst Ising was born in Cologne, Germany, and became US citizen Ernest Ising only after publishing his theory in 1925 and surviving a Nazi concentration camp.) The limit  $Q \to 0$  corresponds to some tree structures (no cyclic links, as in Flory's percolation theory, [7]). Percolation, on the other hand, correponds to the limit  $Q \to 1$ , in the following way: The "free energy" in units of  $k_B T$  is in this limit  $\ln Z = \ln < \exp(N \ln Q) >$   $\ln < \exp[(Q-1)N] >$   $\ln < 1 + (Q-1)N >$  (Q-1)N. Thus for Q near unity this thermal free energy, divided by Q-1, is the number of percolation clusters.

In this way thermal physics and percolation are related, and the cluster numbers N correspond to a free energy. In thermal physics, the negative derivative of the free energy with respect a conjugate field gives the order parameter (e.g magnetic field and magnetisation), and the field derivative of the order parameter is called the susceptibility. For liquid-gas equilibria, the order parameter is the volume (or the density), the field is the pressure (or

chemical potential), and the analog of the susceptibility is the compressibility. This result Eq.(3), not its derivation, we should keep in mind if we now look at the percolation quantities of interest.

Formally we may define for percolation a free energy F as a generating function of a ghost field h:

$$F(h) = \sum_{s} n_s \exp(-hs) \quad . \tag{4}$$

Then its first h-derivative is  $-\sum_s n_s s$ , and the second one is  $\sum_s n_s s^2$ , sums which appear below in the percolation probability  $P_{\infty}$  (the order parameter) and the mean cluster size  $S = \sum_s n_s s^2 / \sum_s n_s s$  (the susceptibility).

# 3 Quantities and Exponents

The basic quantity is  $n_s$ , the number (per site) of clusters containing s sites each, and often is an average over several realizations for the same occupation probability p in the same lattice. Several moments

$$M_k = \sum_{s} n_s s^k \tag{5}$$

are used to define other quantities of interest; in these sums the infinite (spanning) clusters are omitted. The following proportionalities are valid asymptotically in the limit of large lattice size L and for  $p \to p_c$ :

$$F = M_0 \propto |p - p_c|^{2-\alpha} + \dots ; (6a)$$

$$P_{\infty} = p - M_1 \propto (p - p_c)^{\beta} \quad ; \tag{6b}$$

$$S = M_2/M_1 \propto |p - p_c|^{-\gamma} \quad . \tag{6c}$$

Here F is the analog of the thermal free energy, where the three dots represent analytic background terms whose derivatives are all finite. Since every occupied site must belong either to a finite or to an infinite cluster,  $P_{\infty} = p - \sum_{s} n_{s}s$  is the fraction of sites belonging to the infinite cluster and gives the probability that from a randomly selected site we can walk to a lattice boundary along a path of occupied sites. It is thus called the percolation probability but needs to be distinguished from the probability p that a single site is occupied and from the probability R, with  $R(p < p_c) = 0$ ,  $R(p > p_c) = 1$ , that there is a spanning cluster in the lattice.

The quantity S is usually called the mean cluster size, and we follow this tradition even though it is very bad. There are many ways to define a mean size, and polymer chemists have the much more precise notation of a number average  $M_1/M_0$ , a weight average  $M_2/M_1$  and a z average  $M_3/M_2$  for the cluster size (= degree of polymerisation). Physicists arbitrarily call the weight-averaged s the mean cluster size s. Numerically, the exponent s is determined more easily from the "susceptibility" s = s

The radius of a cluster  $R_s$  can be defined as the rms distance  $r_i$ , i = 1, 2, ..., s of cluster sites from the centre of mass  $r_c$  of the cluster (radius of gyration):

$$R_s^2 = <\sum_i (r_i - r_c)^2 / s >$$
 (6a)

where the < ... > are the average over all cluster configurations at probability p. Then the correlation length  $\xi$  is related to the z-average cluster radius through

$$\xi^{2} = \sum_{s} R_{s}^{2} n_{s} s^{2} / \sum_{s} n_{s} s^{2} \propto |p - p_{c}|^{-\nu}$$
 (6b)

with another critical exponent  $\nu$ .

Finally, right at  $p = p_c$ , the cluster numbers decay as

$$n_s \propto 1/s^{2+1/\delta} \tag{7}$$

where  $\delta$  must be positive to allow a finite density  $\sum_{s} n_{s} s = p$ .

These five critical exponents are not independent of each other but are related in d dimensions through the scaling laws:

$$2 - \alpha = \gamma + 2\beta = (\delta + 1)\beta = d\nu \tag{8a}$$

as known from thermal phase transitions; the last equation involving d is not valid in mean field theory (large d) but only for  $d \leq 6$ . Table 2 gives the numerical estimates of the exponents in three dimensions as well as their mean field values for  $d \geq 6$  and their exact two-dimensional results [9, 10]. Thus, for six and less dimensions, if you know two exponents you know them all; thus far.

These scaling laws (8a) can be derived by assuming

$$n_s = s^{-\tau} f[(p - p_c)s^{\sigma}] \quad (\tau = 2 + 1/\delta, \ 1/\sigma = \beta\delta)$$
(8b)

$\overline{d}$	β	$\gamma$	ν
2	5/36	43/18	4/3
3	0.41	1.796	0.88
$\geq 6$	1	1	1/2

Table 2: Critical exponents for percolation clusters. The mean field values are valid for six and more dimensions and also apply to Flory's Bethe approximation and to Erdös-Rényi random graphs. The exponents  $\alpha$ ,  $\delta$ ,  $\sigma$ ,  $\tau$  can be derived from the scaling laws, Eq.(8).

which was first postulated for the thermal Ising model, and then successfully applied to percolation. Here f is a suitable scaling function, which only in the mean-field limit approaches a Gaussian.

For both thermal critical phenomena and percolation, "universality" asserts that these critical exponents are independent of many details and (for the Potts model) depend only on the dimensionality d and the number Q of possible spin states. Since percolation corresponds to  $Q \to 1$  this means that the exponents depend only on d. There are exceptions from this universality for thermal phase transitions, but for random percolation thus far it worked. However, the numerical value of the percolation threshold  $p_c$  is not a critical exponent, depends on the lattice structure, and is different for site and bond percolation.

This universality is one of the reasons why the investigation of exponents is important: They allow to classify models and materials. Similarly, in biology we have many birds of different colours, and many types of domestic animals. Biology became a systematic science only when it was found that all mammals share certain properties, which birds no not have. Thus there is the universality class of mammals.

(The proportionality factors in Eq.(6) are not universal, but some of their combinations are; for example, the ratio of the proportionality factors for S above to below  $p_c$  is universal. In some sense also the probability  $R(p = p_c)$  of a lattice to contain one spanning cluster at the threshold is universal: same for bond and site percolation; however, that probability depends on the boundary conditions and the shape of the sample and thus is far less universal that the mentioned ratio for S.)

Unfortunately, there is another exponent which does not follow from the cluster numbers and radii and for which no scaling law is accepted which relates it to the other exponents above. This refers to the electrical conduc-

tivity

$$\Sigma \propto (p - p_c)^{\mu} \tag{9}$$

when each occupied site (or bond) conducts electrical current and each empty site (or deleted bond) is an insulator. The numerical values are 1.30, 2.0 and 3 in two, three and at least six dimensions. If bonds are related by elastic springs with bending forces, the elastic exponent may be  $\mu + 2\nu$  if entropy effects are negligible, or  $2 - \alpha$  if entropy effects are dominant. Moreover,  $\mu$  is less universal: the above lattice values do not hold on a continuum (conducting spheres which may overlap). Similarly, the kinetics of the Ising model determine a critical exponent which may differ in different variants of the kinetics and may not be related to the static Ising exponents like  $\beta$  and  $\gamma$ .

# 4 Fractal dimension; incipient infinite cluster

### **4.1** Fractal dimension D

Typical objects of geometry classes in school are one-dimensional lines, two-dimensional squares or circles, and three-dimensional cubes or spheres. They have a length (radius) L and a mass (volume for unit density) M with  $M \propto L^d$  for d dimensions. In reality, mother nature produces more more complex objects, like trees, where the mass varies with a power of the tree height below 3:

$$M \propto L^D \quad (D < d, \ L \to \infty).$$
 (10a)

D is the fractal dimension, and such objects are called fractals, particularly if they also are self-similar in that a small twig looks like a big branch, etc. Finite-size scaling theory then relates D of the largest (spanning?) cluster at  $p = p_c$  to the above percolation exponents through

$$D = d - \beta/\nu = (\gamma + \beta)/\nu = 1/(\sigma\nu) = d/(1 + 1/\delta)$$
 (10b)

for  $d \leq 6$ . Thus the critical cluster is about 1.9-dimensional in two and 2.5-dimensional in three dimensions, while in the mean field regime for  $d \geq 6$  we have D = 4.

Why is this so? Any quantity X which is supposed to vary near  $p=p_c$  as  $|p-p_c|^x$  does so only for infinitely large systems. For a finite lattice size L, the

transition is rounded, and neither X nor any of its p-derivatives diverges or becomes exactly zero. In particular, the typical cluster radius or correlation length  $\xi \propto |p-p_c|^{-\nu}$  cannot become infinite but becomes of order L. Then the relation  $X \propto \xi^{-x/\nu}$  is replaced by

$$X(p = p_c) \propto L^{-x/\nu} \tag{11a}$$

at the threshold, and

$$X(p \simeq p_c) = L^{-x/\nu} g[(p - p_c) L^{1/\nu}]$$
(11b)

near the threshold, with a suitable scaling function g. In particular, the fraction  $P_{\infty}$  of sites belonging to the largest cluster at  $p=p_c$  vanishes as  $L^{-\beta/\nu}$ , and the total number M of sites in this cluster as

$$M \propto L^{d-\beta/\nu}$$
 or  $D = d - \beta/\nu$  (11c)

as asserted in Eq.(10b).

Fig.1 shows the second moment  $\chi = M_2 = \sum_s n_s s^2$  in small (curve) and large (+) simple cubic lattices, differing only for  $p \simeq p_c$ .

In a finite lattice, the probability R(p) of a spanning cluster to exist goes from nearly zero to nearly unity in a p-interval proportional to  $1/L^{1/\nu}$ , according to Eq.(11a) with x=0. The derivative  $\mathrm{d}R/\mathrm{d}p$  is the probability that spanning first occurred at probability p. It is plausible that this probability, peaked around  $p_c$ , is a Gaussian, i.e. a normal distribution. Unfortunately, the Evil Empire, also known as the Departments of Chemical Engineering, destroyed [11] this beautiful idea: Since for  $p \simeq p_c$  and  $\xi \sim L$  every part of the lattice is correlated with the rest of the lattice, the central limit theorem does not hold.

(If for  $p \ll p_c$  we let the cluster size s go to infinity, which requires a special algorithm, we get into the universality class of lattice animals, Sec.(2.2). Most simply, in the limit  $p \to 0$ , Eq.(2) simplifies to  $n_s/p^s = g_{st}$ , that means we look at the distribution of configurations with s sites and perimeter t, where all configurations of a given s are weighted equally, whatever their perimeter t is. An important result for these animals is that in three dimensions their radius  $R_s$  varies as  $\sqrt{s}$ , i.e. their fractal dimension is exactly 2. In two dimensions, only numerical estimates exist with  $D \simeq 1.56$ . It is highly unusual that a problem has an exact solution in three but not in two dimensions.)

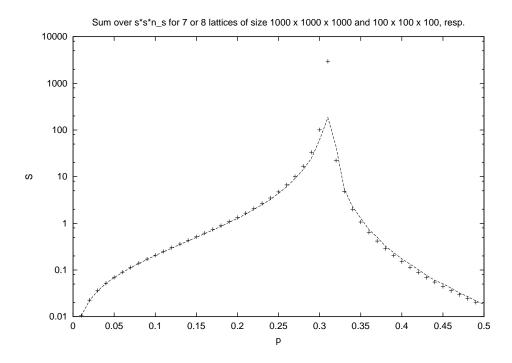


Figure 1: "Susceptibility"  $M_2$  in simple-cubic lattice. For the smaller size the maximum is reduced appreciably.

# 4.2 Incipient infinite cluster

Right at  $p = p_c$  the largest cluster spans the lattice with a pseudo-universal probability  $0 < R(p_c) < 1$ , and then has a density  $P_{\infty}$  going to zero for L going to infinity. It is also called the incipient infinite cluster IIC. Most of the IIC consists of dangling ends which carry no current if the cluster is interpreted as a random resistor network with conductivity  $\Sigma$ , see Eq.(9) above. The remaining current carrying "backbone" has a fractal dimension 1.643 in two dimensions, 1.7 in three and 2 in at least six dimensions and mostly consists of blobs where current flows along several parallel though connected paths. The few "articulation" sites or bonds, the removal of which cuts the network into two or more parts, are also called "red" since all the current flows through them; they have a fractal dimension of only  $1/\nu = 0.75$ , 1.14 and 2 in two, three and > six dimensions.

How many infinite clusters do we have? The easy answer is: none below, perhaps one at and always one above  $p_c$  in an infinite network. Indeed,

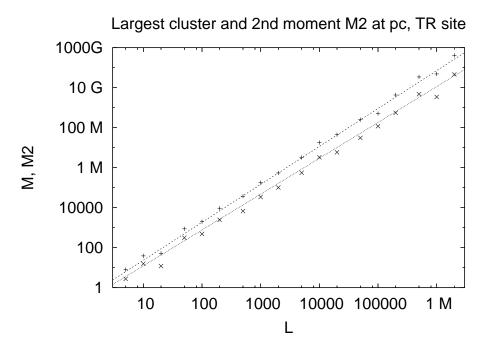


Figure 2: Number M of sites in largest cluster (+) and susceptibility  $M_2$  (x) at  $p=p_c=1/2$  for triangular site percolation. The two straight lines have the exact slopes D=91/38 and  $\gamma/\nu=43/24$  predicted by finite-size scaling. The largest lattice took about 36 hours on a workstation with 2 Gigabytes memory. Tiggemann [6] simulated  $L=7\times10^6$ , 25024, 1305, 225 for d=2, 3, 4, 5 on a large parallel computer.

this is what was claimed mathematically in the 1980's [12]: The number of infinite clusters is zero, one or infinite. Later mathematics excluded the last choice of infinitely many clusters, even though in seven dimensions scaling arguments, confirmed by numerical studies [13], indicated the number of IIC to go to infinity for increasing L in seven dimensions. Only in 1995 and later Aizenman [14] predicted that in all dimensions one may have several spanning clusters at  $p = p_c$ , in agreement with simulations [15].

Why were the earlier uniqueness theorems irreproducible at  $p_c$  and for very elongated rectangles even above  $p_c$  [16]? A clear definition of "infinite" is missing in some of the mathematics, although [12] defined a cluster as infinite if its cardinality (= number of sites in it) is infinite for  $L \to \infty$  in a hypercubic

lattice of  $L^d$  sites. Clear definitions of infinity are, of course, needed for reliable proofs [17]. Measure theory as applied in some theorems may be based on some axioms which are not applicable for a fractal IIC. Very simply, imagine each line of an  $L \times L$  square lattice to have one randomly selected site occupied and all others empty. The set of occupied site then has cardinality L which is infinite for infinite lattices, but its density becomes zero. Does your measure theory agree with this? More relevant for percolation, even for  $p < p_c$  the largest cluster has a size increasing logarithmically with lattice size and thus can be described as infinite, invalidating the percolation threshold as the onset of infinite clusters. Thus infinite might be defined as increasing with a positive power of L, i.e. having a positive fractal dimension. Then we have infinitely many infinite clusters only at  $p = p_c$ , though in most cases only the largest of them is a spanning cluster. Using "spanning" as a definition of an infinite cluster seems to cause the smallest problems.

Thus one should not regard a question as settled if some mathematical theorem claims to have answered it. The mathematics may not apply to the same problem one is interested in, or (see bootstrap percolation in this encyclopedia) may apply only for unrealistically large lattices. On the other hand, also computer simulations should be relied upon only if confirmed independently. And in the interpretation of simulation results one should be objective and not try to agree with prevailing theories. For example, [13] might already have seen the multiplicity of infinite clusters in five dimensions, not only in seven, had she not followed her obviously incompetent postdoctoral mentor.

(On a more positive side, mathematicians [18] solved biased diffusion on percolating clusters above  $p_c$  only a few years after physicists still had controversies about their simulations.)

# 5 Simple Renormalisation Group

Why are scaling laws and finite-size scaling so simple? Why is universality valid for the exponents? These question arose for thermal critical phenomena as well as for percolation. The main reason is that the correlation length  $\xi$  goes to infinity at the critical point. Thus all approximations which restrict the correlation to some finite lengths eventually become wrong, and instead the scaling ideas become correct. They were explained by Ken Wilson through what he called renormalisation group, around 1970, and he got

the physics Nobel prize for it in 1982. Basically, since correlations extend over long distances, the single atom or lattice point becomes irrelevant and can be averaged over. In politics, we have a similar effect: Many democracies are based on electoral districts, and the candidate winning most votes within this district represents this district in the national parliament. It is the cooperation of many people within the electoral district, not the single vote, which is decisive.

Returning to an  $L \times L$  lattice, we can divide it into many blocks of linear dimension b, and treat a block analogously to an electoral district. Thus in an Ising model, if the majority of block spins point upward, the whole block is represented by a superspin pointing up, analogous to the single representative in politics. These block spins then act like the original spins, one can put  $b \times b$  superspins into one superblock, and have just one super-representative following the majority opinion of the representatives within the superblock. This process can be continued: at each stage  $b \times b$  lower representatives are renormalised into a single higher representative.

Such a renormalisation by majority rule works fine with Ising spins, but percolation deals with connections, not with up and down spins. Thus for percolation a  $b \times b$  block is renormalised into an occupied supersite if and only if there is a spanning cluster within the block; otherwise the superblock is defined empty. In this way, whole blocks are renormalised into singe sites via connectedness. And the renormalisation is reduced to the standard question which was asked already before Wilson's invention: Does a  $b \times b$  lattice have a spanning cluster? The supersite is thus occupied if and only if the block spans, which happens with probability  $R_b(p)$ . If we call the p' probability of the supersite to be occupied, we thus have

$$p' = R_b(p) \quad . \tag{12a}$$

If we are at  $p = p_c$ , then the renormalisation should not change anything drastic since  $\xi$  is larger than any b; thus if the renormalisation would be exact we would have

$$p_c' = R_b(p_c) \quad . \tag{12b}$$

Practically we determine a fixed point  $p = p^*$  such that

$$p^* = R_b(p^*) \quad . \tag{12c}$$

and then find  $p_c$  as the limit of  $p^*$  for  $b \to \infty$ , which again is similar what percolation experts did before renormalisation theory.

A particularly simple example is the triangular site percolation problem with  $p_c = 1/2$ , if we do not divide the lattice into large  $b \times b$  blocks but into small triangles of three sites which are nearest neighbours, as shown on the left:

The triangle contains a spanning cluster if either all three sites are occupied (x, central diagram) or two sites are occupied (x) and one site is empty (, right diagram). The first choice appears with probability  $p^3$ , the second with probability  $p^2(1-p)$ . However, this second choice has three possible orientations since each of the three sites can be the single empty site. Thus the total probability of the triangle to have a spanning cluster is

$$p' = p^3 + 3(1-p)p^2 (13a)$$

with three fixed points  $p^*$  where p' = p:

$$p^* = 0, \quad p^* = 1/2, \quad p^* = 1 \quad .$$
 (13b)

The second of these fixed points is the percolation threshold, while the first correponds to lattice animals (section 2.2 and end of section 4.1) and the third to compact non-fractal clusters. With somewhat more effort one can derive also a good approximation for  $\nu$ .

This agreement of the fixed point  $p^*$  with the true threshold  $p_c = 1/2$  is not valid for other lattices or block choices. Nevertheless there was a widespread fixed-point consensus that  $R_b(p_c) = p_c$  for sufficiently large b. Regrettably, the Evil Empire [19] again destroyed this beauty and found  $R_b(p_c) = 1/2$  for square site percolation where  $p_c \simeq 0.593$ . In general,  $R(p_c)$  is a pseudo-universal quantity depending on boundary conditions and sample shape, while  $p_c$  for large samples is independent of these details but is different for site and bond percolation and depends on the size of the neighbourhood. Life was much nicer before. Fortunately, if a fixed point is determined by Eq.(12c) and the block size goes to infinity, then the fixed point still approaches  $p_c$ .

### 6 Future Directions

This review summarised the basic theory, particularly when it was not yet contained in the earlier books [22]. Applications were left to the Sahimi book [22]; even for the very first application [1] there is not yet a complete consensus that the three-dimensional percolation exponents apply to polymer gelation. More recent applications are social percolation [20] for marketing by word-of-mouth, and stock market fluctuations due to herding among traders [21].

Percolation theory, similar to Fortran programming or capitalism, was thought to be finished but seems to be alive and kicking. Nevertheless I think the future is more in its applications.

The manuscript was improved by criticism of A. Aharony.

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